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Abstract. q-integration is defined for the complex quantum plane. The existence of classical and Grassmanian limits of q-integration is proved. Quantum versions of Cauchy's and Stokes' theorems are formulated.

1. Introduction

The concept of integration plays a very important role in mathematical physics, the theory of group representation, and field theory. In the case of quantum groups (Hopf algebras and C^*-algebras) the problem was intensively investigated by pure algebraists [9, 10], mathematical physicists [7, 8] and physicists [3]. From the algebraic point of view [7-10], integration connected with the invariant measure (Haar measure) is the most interesting. However, at the Hopf algebra level the existence of the invariant measure is proved only for commutative (co-semisimple) and for finite-dimensional Hopf algebras which are useless in the construction of quantum groups. At the C^*-algebra level, Woronowicz [8] proved the existence of the Haar measure for compact quantum groups.

In this paper we deform Riemann's integral on the quantum complex plane (which is a non-compact, non-commutative, infinite-dimensional algebra). We show that our deformation is continuous with respect to the changing of deformation parameters. We also show that the classical Riemann's integral and Grassmanian Berezin's integral are two limit cases of our deformation.

In the last section of this paper we discuss briefly some properties of q-integrals, i.e. we show that complex q-integrals fulfil Stokes' and Cauchy's theorems. Finally, we define the deformed (quantum) square measure.

2. Differential calculus on Manin's plane

It is known [1] that one can construct two distinct families of differential calculus on Manin's plane. The quantum (Manin's) plane $\mathbb{C}_q^{2|0}$ [2] is a graded module over $\mathbb{C}$ generated by two elements x, y obeying the defining relation $xy = qyx$, where q is a parameter. It has a bialgebra structure, given by the relations $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes 1 + x \otimes y$, $\varepsilon(x) = 1$, $\varepsilon(y) = 0$, where Δ is the usual coproduct and ε is the counit. To get a differential calculus we introduce a linear differential operator d, which is nilpotent and obeys the Leibniz rule with gradation, and we demand the differential calculus to be scale invariant. The full algebra

$$\mathscr{A} = \Lambda^0\mathscr{A} \oplus \Lambda^1\mathscr{A} \oplus \Lambda^2\mathscr{A} \qquad \text{where} \qquad \Lambda^0\mathscr{A} = \mathbb{C}_q^{2|0},\ \Lambda^0\mathscr{A} \xrightarrow{d} \Lambda^1\mathscr{A} \xrightarrow{d} \Lambda^2\mathscr{A}$$

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is then an associative graded algebra generated by x, y and dx, dy. We assume that $dx \wedge dx$ as well as $dy \wedge dy$ is equal to zero. We also define (right) partial derivatives in the directions x and y of any function $f(x, y) \in \mathbb{C}_q^{2|0}$ ($f(x, y)$ is understood as a formal power series of variables x and y with coefficients from $\mathbb{C}$) by the equation

$$df(x, y) = dx\, D_x f(x, y) + dy\, D_y f(x, y).$$

With these assumptions, using the consistency relations [5] and assuming that the equation $df(x, y) = 0$ implies $f(x, y) = c$ ($c \in \mathbb{C}$), we get two families of differential calculus:

Family I,

$$\begin{aligned}
& xy = qyx \\
& x\, dx = p\, dx\, x \qquad x\, dy = q\, dy\, x \\
& y\, dx = q^{-1}\, dx\, y \qquad y\, dy = r\, dy\, y \\
& dx \wedge dy = -q\, dy \wedge dx \\
& D_x x = 1 + pxD_x \qquad D_y x = qxD_y \qquad D_x y = q^{-1} yD_x \\
& D_y y = 1 + ryD_y \qquad D_x D_y = qD_y D_x
\end{aligned}$$

where $p, q, r \in \mathbb{C}$ and

$$D_x f(x, y) = \frac{1}{x} \frac{f(px, y) - f(x, y)}{p - 1}$$

$$D_y f(x, y) = \frac{1}{y} \frac{f(x, ry) - f(x, y)}{r - 1}.$$

Family II,

$$\begin{aligned}
& xy = qyx \\
& x\, dx = s\, dx\, x \qquad x\, dy = (s-1)\, dx\, y + q\, dy\, x \qquad y\, dx = sq^{-1}\, dx\, y \\
& y\, dy = s\, dy\, y \qquad dx \wedge dy = -qs^{-1}\, dy \wedge dx \\
& D_x x = 1 + sxD_x + (s-1)yD_y \qquad D_x y = qxD_y \qquad D_x y = sq^{-1} yD_x \\
& D_y y = 1 + syD_y \qquad D_x D_y = qs^{-1} D_y D_x
\end{aligned}$$

where $q, s \in \mathbb{C}$ and

$$D_x f(x, y) = \frac{1}{x} \frac{f(sx, sy) - f(x, sy)}{s - 1}$$

$$D_y f(x, y) = \frac{1}{y} \frac{f(x, sy) - f(x, y)}{s - 1}.$$

An immediate possibility of introducing a complex structure in Manin's plane lies in putting $z = x + iy$, $\bar{z} = x - iy$, and making all parameters real. This converts Manin's plane into the complex quantum plane $\mathbb{C}_q$. It is now possible to introduce a set of quantum holomorphic functions $\mathscr{H}(\mathbb{C}_q)$, given by the quantized Cauchy–Riemann's equation $D_x H(x, y) = -iD_y H(x, y)$ for any $H(x, y) \in \mathscr{H}(\mathbb{C}_q)$. This equation is explicitly solved in both cases in [1]. Solutions of Cauchy–Riemann's equations show that quantum holomorphicity does not mean that the holomorphic function can be represented as the formal power series of one variable z. Moreover, the set $\mathscr{H}(\mathbb{C}_q)$ is an algebra only in the first case when we put $p = q$ and $r = q^{-1}$.

There exists the possibility of another definition of the complex structure on Manin's plane [1, 12]. We observe that $\mathbb{C}_q^{2|0}$ admits $*$-algebra structure ($*$ is an antilinear involution) if q is real and $x^* = y$. The algebra $\mathscr{A}$ becomes an involutive algebra when it is defined by the multiplication rules I, p is real and $r = p^{-1}$. This observation suggests the consideration of the quantum complex variable $\zeta = x$, and its complex (Hermitian) conjugation $\zeta^* = y$. (Note that this $*$-structure is not compatible with the coproduct Δ, defined above—compare, for example, [11]. In fact $\mathbb{C}_q^*$ is a quantum space and does not demand bialgebra structure.) Now the equations (family I) take the following form:

$$\begin{aligned}
&\zeta\zeta^* = q\zeta^*\zeta \\
&\zeta\, \mathrm{d}\zeta = p\, \mathrm{d}\zeta\, \zeta \qquad \zeta\, \mathrm{d}\zeta^* = q\, \mathrm{d}\zeta^*\, \zeta \qquad \mathrm{d}\zeta \wedge \mathrm{d}\zeta^* = -q\, \mathrm{d}\zeta^* \wedge \mathrm{d}\zeta \\
&D_\zeta\zeta = 1 + p\zeta D_\zeta \qquad D_\zeta\zeta^* = q^{-1}\zeta^* D_\zeta \qquad D_\zeta D_{\zeta^*} = qD_{\zeta^*}D_\zeta \\
&D_\zeta\, \mathrm{d}\zeta = p^{-1}\, \mathrm{d}\zeta\, D_\zeta \qquad D_\zeta\, \mathrm{d}\zeta^* = q^{-1}\, \mathrm{d}\zeta^*\, D_\zeta \\
&(\mathrm{d}\zeta)^* = \mathrm{d}\zeta^* \qquad (D_\zeta)^* = -p^{-1}D_{\zeta^*}.
\end{aligned} \tag{1}$$

In this paper we shall deal with this last definition of the quantum complex plane—$\mathbb{C}_q^*$ ($\mathbb{C}_q^*$ is also called [6] an algebra of polynomials on the Euclidean quantum plane). $\mathbb{C}_q^*$ has some interesting properties. First of all, the set of all holomorphic functions $\mathfrak{H}(\mathbb{C}_q^*)$, forms an algebra and every holomorphic function can be written as a formal power series of one variable ζ. This last statement follows immediately from the definition of a holomorphic function (a function $h(\zeta, \zeta^*) \in \mathbb{C}_q^*$ is holomorphic if $\mathrm{d}h(\zeta, \zeta^*) = \mathrm{d}\zeta\, h(\zeta, \zeta^*)$ i.e. $D_{\zeta^*}h(\zeta, \zeta^*) = 0$.

3. The *q*-integration

In this section we use the following notation: $f(\zeta) = \Sigma f_n\zeta^n$, $f(\zeta^*) = \Sigma f_n\zeta^{*n}$, where $f_n \in \mathbb{C}$.

Let us look first for the possibility of a definition of the inverse operation for the derivative D_ζ, i.e. the $\mathbb{C}$-linear operation $\int^{p,q} \mathrm{d}\zeta\colon \mathbb{C}_q^*, \to \mathbb{C}_q^*$, such that $D_\zeta\{\int^{p,q} \mathrm{d}\zeta\, f(\zeta)\} = f(\zeta)$ (analogously we can look for the $\mathbb{C}$-linear operation $\int^{p,q} \mathrm{d}\zeta^*\colon \mathbb{C}_q^*, \to \mathbb{C}_q^*$ such that $D_{\zeta*}\{\int^{p,q} \mathrm{d}\zeta^* f(\zeta^*)\} = f(\zeta^*)$). It is enough to restrict ourselves to the monomials ζ^n, (ζ^{*n}), $n \in \mathbb{N}$, and then we find immediately that

$$\begin{aligned}
&\int^{p,q} \mathrm{d}\zeta\, \zeta^n = \frac{\zeta^{n+1}}{[n+1]_p} + \text{constant} \\
&\int^{p,q} \mathrm{d}\zeta^*\, \zeta^{*n} = p^n \frac{\zeta^{*n+1}}{[n+1]_p} + \text{constant}
\end{aligned} \tag{2}$$

where $[n]_p \equiv (1 - p^n)/(1 - p)$.

A similar kind of integration for a *commutative* variable with the deformed derivative (Gauss' (Jackson's) derivative, which is simply a consequence of the definition of the differential calculus in our case) was introduced in [3]. However, the integration considered in [3] loses its meaning for $p \to -1$, which is simply related to the fact that the inverse operation to the Grassmanian derivative does not exist. At least for this reason it is more convenient to define an analogue of the definite integral, i.e. a linear, continuous functional over elements of our involutive algebra. We propose the following definition:

$$\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta\,\zeta^n = (\beta^{n+1}-\alpha^{n+1})\frac{[(p+1)/2|p|^{1/2}]^{(n+1)/2}}{[n+1]_p} \tag{3}$$

$$\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta^*\,\zeta^{*n} = (\beta^{n+1}-\alpha^{n+1})p^n\frac{[(p+1)/2|p|^{1/2}]^{(n+1)/2}}{[n+1]_p} \tag{4}$$

where $\alpha, \beta \in \mathbb{C}$. Functional $\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta\,\zeta^n$ can be interpreted as the first of the integrals (2) taken in the limits $\beta[(1+p)/2|p|^{1/2}]^{1/2}$, $\alpha[(1+p)/2|p|^{1/2}]^{1/2}$. We can also observe that

$$\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta^*\,\zeta^{*n} = \int_{[\alpha,\beta]}^{p^{-1}q} \mathrm{d}\zeta\,\zeta^n \tag{5}$$

$$\int_{[\alpha,\beta]}^{q} \mathrm{d}\zeta\,\zeta^n = \lim_{p\to 1}\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta\,\zeta^n = \frac{\beta^{n+1}-\alpha^{n+1}}{n+1} \tag{6}$$

$$\int_{[\alpha,\beta]}^{-1-1} \mathrm{d}\zeta\,\zeta^n = \lim_{\substack{p\to -1\\ q=-1}}\int_{[\alpha,\beta]}^{p,q} \mathrm{d}\zeta\,\zeta^n = \begin{cases}(\beta^2-\alpha^2)/2 & \text{for } n=1\\ 0 & \text{for } n\neq 1.\end{cases} \tag{7}$$

Equations (6) and (7) show that the q-integral is equivalent to Riemann's integral if $p=1$, and that it behaves like Berezin's integral if $p=-1$. For this reason we can define a class of definite integrals taking as limits (see (3) and (4)) $\beta[(1+p)/2]^{1/2}\gamma_1(p)$, $\alpha[(1+p)/2]^{1/2}\gamma_2(p)$, where $\gamma_i(1)=1$ and $\gamma_i(-1)=1$, $i=1,2$. This is strictly related to the fact that we cannot put β as a q-variable and then interpret a definite q-integral as a q-function. We put $\gamma_i(p)=|p|^{1/2}$ for simplicity.

Note also that our integral, except for linearity, satisfies

$$\int_{[\alpha,\beta]}^{p,q}[\cdot]+\int_{[\beta,\gamma]}^{p,q}[\cdot]=\int_{[\alpha,\gamma]}^{p,q}[\cdot] \qquad \int_{[\alpha,\alpha]}^{p,q}[\cdot]=0 \tag{8}$$

for any $\alpha, \beta, \gamma \in \mathbb{C}$, so

$$\int_{[\alpha,\beta]}^{p,q}[\cdot] = -\int_{[\beta,\alpha]}^{p,q}[\cdot].$$

It is easy to check that the $*$-operation is in agreement with (3) and (4), i.e.

$$\left(\int_{[\alpha,\beta]}^{p,q}\mathrm{d}\zeta\,\zeta^n\right)^* = \int_{[\alpha,\beta]}^{p,q}(\mathrm{d}\zeta\,\zeta^n)^*. \tag{9}$$

With these preliminary definitions we can start to construct a q-like integral along the path. From now on we will reserve the symbol z for the complex variable, and the symbol ζ for the q-complex variable. Let Γ be any path in $\mathbb{C}$. We propose the following definition of the q-integral 'along the path Γ':

$$\int_{\Gamma}^{p,q}\mathrm{d}\zeta\,\zeta^n\zeta^{*m} = q^{m(n+1)/2}\frac{n+1}{[n+1]_p}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{m/2}\int_{\Gamma}\mathrm{d}z\,z^n\bar{z}^m \tag{10}$$

$$\int_{\Gamma}^{p,q} d\zeta^*\,\zeta^{*n}\zeta^m = q^{-m(n+1)/2}\frac{n+1}{[n+1]_{1/p}}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{m/2}\int_{\Gamma} d\bar{z}\,\bar{z}^n z^m. \tag{11}$$

Let D now be a domain in $\mathbb{C}$. We define

$$\int_{D}^{p,q} d\zeta\,\zeta^n \wedge d\zeta^*\,\zeta^{*m}$$

$$= q^{(m+1)(n+1)/2}\frac{n+1}{[n+1]_p}\frac{m+1}{[m+1]_{1/p}}$$

$$\times\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(m+1)/2}\int_{D} dz\wedge d\bar{z}\,z^n\,\bar{z}^m. \tag{12}$$

The definitions above suggest the following.

Prescription for q-integration:

(i) Integration along a path.

Take any quantum one-form $d\zeta\,f(\zeta,\zeta^*)=\Sigma_{n,m} f_{nm}\,d\zeta\,\zeta^n\zeta^{*m}$, and a path $\Gamma\subset\mathbb{C}$. Define the classical one-form,

$$dz\,F(z)=\sum_{n,m} q^{m(n+1)/2}\frac{n+1}{[n+1]_p}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{m/2} f_{nm}\,dz\,z^n\bar{z}^m \tag{13}$$

and finally calculate $\int_\Gamma^{p,q} d\zeta\,f(\zeta,\zeta^*)=\int_\Gamma dz\,F(z)$.

(ii) *-integration along a path.

Take any quantum one-form $d\zeta^*\,f(\zeta^*,\zeta)=\Sigma_{n,m} f_{nm}\,d\zeta^*\,\zeta^{*n}\zeta^m$, and a path $\Gamma\subset\mathbb{C}$. Define the classical one-form,

$$d\bar{z}\,F(\bar{z})=\sum_{n,m} q^{-m(n+1)/2}\frac{n+1}{[n+1]_{1/p}}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{m/2} f_{nm}\,d\bar{z}\,\bar{z}^n z^m$$

and finally calculate $\int_\Gamma^{p,q} d\zeta^*\,f(\zeta^*,\zeta)=\int_\Gamma d\bar{z}\,F(\bar{z})$.

(iii) Integration in a domain.

Take any quantum two-form $d\zeta\wedge d\zeta^*\,g(\zeta,\zeta^*)=\Sigma_{n,m} g_{nm}\,d\zeta^n\,\zeta^{*m}$ and a domain $D\subset\mathbb{C}$. Define the classical two-form,

$$dz\wedge d\bar{z}\,G(z,\bar{z})$$

$$= q^{n+(m+1)(n+1)/2}\frac{n+1}{[n+1]_p}\frac{m+1}{[m+1]_{1/p}}$$

$$\times\left(\frac{1+p}{2|p|^{1/2}}\right)^{(n+1)/2}\left(\frac{1+p}{2|p|^{1/2}}\right)^{(m+1)/2} g_{nm}\,dz\wedge d\bar{z}\,z^n\bar{z}^m$$

and finally calculate $\int_D^{p,q} d\zeta\wedge d\zeta^*\,g(\zeta,\zeta^*)=\int_D dz\wedge d\bar{z}\,G(z,\bar{z})$.

Finally we want to note that the integration defined above obeys *Stokes' theorem*, i.e. if $D\subset\mathbb{C}$ is a domain, ∂D is a boundary of D and $\omega(\zeta,\zeta^*)$ is a one-form, then

$$\int_{\partial D}^{p,q}\omega(\zeta,\zeta^*)=\int_{D}^{p,q} d\omega(\zeta,\zeta^*). \tag{14}$$

4. Quantum Cauchy's theorems

Let C_a be a circle $z\bar{z}=a$, and $p>0$, then

$$\int_{C_a}^{p,q} \mathrm{d}\zeta\, \zeta^n \zeta^{*m} = \begin{cases} 2\pi q^{(n+1)^2/2} p^{-(n+1)/2} \left(\dfrac{p+1}{2}\right)^{n+1} \dfrac{n+1}{[n+1]_p} a^{n+1}\mathrm{i} & \text{for } m=n+1 \\ 0 & \text{for } m \neq n+1 \end{cases} \tag{15a}$$

$$\int_{C_a}^{p,q} \mathrm{d}\zeta^*\, \zeta^{*n} \zeta^{m} = \begin{cases} -2\pi q^{-(n+1)^2/2} p^{(n-1)/2} \left(\dfrac{p+1}{2}\right)^{n+1} \dfrac{n+1}{[n+1]_p} a^{n+1}\mathrm{i} & \text{for } m=n+1 \\ 0 & \text{for } m \neq n+1. \end{cases} \tag{16a}$$

For the special case $n+1=m=1$ and $a=1$ these equations reduce to

$$\int_{C_1}^{p,q} \mathrm{d}\zeta\, \zeta^* = 2\pi(q/p)^{1/2}[(1+p)/2]\mathrm{i} \tag{15b}$$

$$\int_{C_1}^{p,q} \mathrm{d}\zeta^*\, \zeta = -2\pi(q/p)^{-1/2}[(1+p^{-1})/2]\mathrm{i}. \tag{16b}$$

Equations (13) and (15a) give the following *Cauchy's q-integral formula*:

$$F(a) = \frac{[q(p+1)/2|p|^{1/2}]^{-1/2}}{2\pi\mathrm{i}r^2} \int_\Gamma^{p,q} \mathrm{d}\zeta\, \zeta^* f(q^{1/2}\zeta) \tag{17}$$

where Γ is a circle $|z-a|=r$ and $f(\zeta)$ is a quantum holomorphic function such that the function $F(a)$ defined by (13) ($m=0$) is holomorphic in a simply connected open domain $\Omega \supset \Gamma$.

Now it is possible to give the quantized version of (the local) *Cauchy's theorem.* Let $f(\zeta)$ be any quantum holomorphic function, $f(\zeta)=\Sigma_{n=0} f_n\zeta^n$, and Ω be a domain in $\mathbb{C}$ such that $F(z)$ (given by (13)) is continuous in Ω and the series

$$\sum_{n=1} \frac{1}{[n]_p} \left(\frac{1+p}{2|p|^{1/2}}\right)^{n/2} f_{n-1} z^n \tag{18}$$

is absolutely convergent in Ω. Then

$$\int_\Gamma^{p,q} \mathrm{d}\zeta f(\zeta) = 0 \tag{19}$$

for any closed road $\Gamma \subset \Omega$.

Using the prescription for q-integration (13) one can easily give a quantum version of *Cauchy's global theorem*, i.e. if $f(\zeta)$ is a quantum holomorphic function and $\Omega \subset \mathbb{C}$ is a simply connected domain such that the function $F(z)$ (given by (13)) is analytic in $\mathrm{cl}(\Omega)$, then

$$\int_\Gamma^{p,q} \mathrm{d}\zeta f(\zeta) = 0 \tag{20}$$

for any closed road $\Gamma \subset \Omega$.

Finally, we can define the quantum analogue of the square measure of a (flat) domain. Let $D \subset \mathbb{C}$ be a domain. The number

$$|D|_{pq} = \tfrac{1}{2}\left| \int_D^{p,q} \mathrm{d}\zeta \wedge \mathrm{d}\zeta^* \right| \tag{21}$$

is called a *quantum square measure* of D.

Let us briefly discuss the invariance of the measure $|\cdot|_{pq}$. Consider the quantum group of motions on Manin's plane $\mathrm{ISO}(2)_q$ [12] generated by the matrix

$$\begin{bmatrix} a & 0 & u \\ 0 & a^* & u^* \\ 0 & 0 & I \end{bmatrix} \tag{22}$$

with defining relations

$$aa^* = a^*a = 1 \qquad uu^* = qu^*u \qquad au = qua \tag{23}$$

and natural bialgebra structure

$$\begin{aligned} &\Delta(\underline{a}) = \underline{a} \otimes \underline{a} && \Delta(\underline{u}) = \underline{a} \otimes \underline{u} + \underline{u} \otimes \underline{I} \\ &\varepsilon(a) = 1 && \varepsilon(u) = 0 \\ &S(a) = a^* && S(u) = -a^*u. \end{aligned} \tag{24}$$

If we define the following comodule action $\delta : \mathbb{C}_q^* \to \mathrm{ISO}(2)_q \otimes \mathbb{C}_q^*$

$$\delta\left(\begin{bmatrix} \zeta \\ \zeta^* \\ I \end{bmatrix}\right) = \begin{bmatrix} a & 0 & u \\ 0 & a^* & u^* \\ 0 & 0 & I \end{bmatrix} \otimes \begin{bmatrix} \zeta \\ \zeta^* \\ I \end{bmatrix} \tag{25}$$

then we can obtain that $|\cdot|_{pq}$ is invariant under action of δ.

It is easy to verify that $|D|_{pq} = |q/p|^{1/2}[(1+p)/2]|D|$ where $|D|$ denotes a classical square measure of D.

5. Conclusions

We have shown that it is possible to construct the quantum analogue of a classical complex integral, which has classical and Grassmannian limits. Our definition was motivated by practical reasons, and it can be used in the analysis of coherent states, and in the construction of Bargmann's space. This work is supported by KBN Grant 2 0218 91 01.

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